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Problems based on Cayley's theorem.

Q.1. If H be a subgroup of group G and
$$T = \{x \in G : xH = Hx\}.$$
Show that T is a subgroup of G.

Soln: Since $e \in G$ and $eH = He \Rightarrow e \in T$

$\Rightarrow T$ is a non-empty subset of G.

Let $x_1, x_2 \in T$ so that $x_1 H = Hx_1$, $x_2 H = Hx_2$

Now, $x_2 \in T \Rightarrow x_2 H = Hx_2$
$$\Rightarrow x_2^{-1}(x_2 H)x_2^{-1} = x_2^{-1}(Hx_2)x_2^{-1}$$
$$\Rightarrow x_2^{-1}x_2(Hx_2^{-1}) = (x_2^{-1}H)(x_2 x_2^{-1})$$
$$\Rightarrow e(Hx_2^{-1}) = (x_2^{-1}H)e$$
$$\Rightarrow Hx_2^{-1} = x_2^{-1}H \Rightarrow x_2^{-1} \in T$$

Thus, $x_2 \in T \Rightarrow x_2^{-1} \in T$.

Also, $(x_1 x_2^{-1})H = x_1(x_2^{-1}H) = x_1(Hx_2^{-1}) = (x_1 H)x_2^{-1} = (Hx_1)x_2^{-1} = H(x_1 x_2^{-1})$

$\Rightarrow x_1 x_2^{-1} \in T$.

Thus, T is a non-empty subset of G and $x_1, x_2 \in T$

$\Rightarrow x_1 x_2^{-1} \in T$.

$\therefore$ T is a subgroup of G.

Q.2. Find the regular permutation group isomorphic to the multiplicative group $G = \{1, -1, i, -i\}$

Soln: We know by Cayley's theorem the regular permutation group G' isomorphic to G consist the following four permutations f_1, f_2, f_3, f_4.

$$f_1 = \begin{pmatrix} 1 & -1 & i & -i \\ 1\cdot 1 & 1\cdot(-1) & 1\cdot i & 1\cdot(-i) \end{pmatrix} = \begin{pmatrix} 1 & -1 & i & -i \\ 1 & -1 & i & -i \end{pmatrix} = I$$

$$f_2 = \begin{pmatrix} 1 & -1 & i & -i \\ (-1)\cdot 1 & (-1)(-1) & (-i)\cdot i & (-i)(-i) \end{pmatrix} = \begin{pmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{pmatrix} = (1, -1)(i, -i)$$

$$f_3 = \begin{pmatrix} 1 & -1 & i & -i \\ i\cdot 1 & i\times(-1) & i\cdot i & i(-i) \end{pmatrix} = \begin{pmatrix} 1 & -1 & i & -i \\ i & -i & -1 & 1 \end{pmatrix} = (1, i, -1, -i)$$

$$f_4 = \begin{pmatrix} 1 & -1 & i & -i \\ (-i)\cdot 1 & (-i)(-i) & (-i)\cdot i & (-i)(-i) \end{pmatrix} = \begin{pmatrix} 1 & -1 & i & -i \\ -i & i & 1 & -1 \end{pmatrix} = (1, -i, -1, i)$$

Solved.

Q.3. Prove that these elements of a group G which commute with the square of a given element b of G from a subgroup of G.

Soln: Let $H = \{x \in G : xb^2 = b^2 x\}$.

Since, $b^2 \in G$ and $eb^2 = b^2 e$, so, $e \in H \Rightarrow H$ is a non-empty subset of G.

Let $x_1, x_2 \in H$ so that $x_1 b^2 = b^2 x_1$ and $x_2 b^2 = b^2 x_2$

Now, $(x_1 x_2) b^2 = x_1 (x_2 b^2)$ [by associativity]

$= x_1 (b^2 x_2)$ [$\because x_2 b^2 = b^2 x_2$]

$= (x_1 b^2) x_2$ [by associativity]

$= (b^2 x_1) x_2 = b^2 (x_1 x_2)$ [$\because x_1 b^2 = b^2 x_1$]

$\Rightarrow x_1, x_2 \in H \Rightarrow x_1 \cdot x_2 \in H$ ——— (I)

Also, $x_1 \in H \Rightarrow x_1 b^2 = b^2 x_1$

$\Rightarrow x_1^{-1} (x_1 b^2) x_1^{-1} = x_1^{-1} (b^2 x_1) x_1^{-1}$

$\Rightarrow (x_1^{-1} x_1)(b^2 x_1^{-1}) = (x_1^{-1} b^2)(x_1 x_1^{-1})$

$\Rightarrow e (b^2 x_1^{-1}) = (x_1^{-1} b^2) e \Rightarrow b^2 x_1^{-1} = x_1^{-1} b^2$

Thus, $x_1 \in H \Rightarrow x^{-1} \in H$ ——— II

Now, from (I) and (II), we conclude that H is a subgroup of G.

Proved.